PHYS 301 – Assignment #4

Due Wednesday, Nov. 20 at 14:00

Imagine a uniformly-charged sphere spinning with angular speed ω about an axis that passes through its centre. Because the sphere is charged, its spin results in the motion of charge or a current that creates a magnetic field. In this assignment, we will attempt to calculate the magnetic field produced at the centre of a charged sphere spinning with angular speed ω. Along the way, we will also find the magnetic fields due to a spinning loop of charge and a spinning charged disk. Next, we will deduce the magnetic moments of spinning charged objects by relating them to the corresponding angular momenta of the spinning masses. Finally, we will conclude by making some comments about the intrinsic "spin angular momentum" and magnetic moment of elementary particles, like the electron.

1(a) Consider a sphere of total charge Q and radius R. The sphere is uniformly-charged with charge density ρ and rotates with angular velocity ω about an axis that passes through its centre. In spherical coordinates, a volume element in the sphere is given by:

$$\mathrm{d}\tau = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\tau.$$

Ultimately, we will be using the Biot-Savart law:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I \, \mathrm{d}\boldsymbol{\ell} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^{\,2}}$$

to calculate the magnetic field at the centre of the of the rotating sphere. Show that, for our volume element, the quantity $I \, \mathrm{d}\ell$ can be expressed as:

$$I \, \mathrm{d}\ell = \rho\omega r^3 \sin^2\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi.$$

(b) Next, using the Biot-Savart law, show that the magnetic field at the centre of the sphere due to just the volume element from part (a) would be given by:

$$dB = \frac{\mu_0}{4\pi}\,\rho\omega r\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{1}$$

Recall that $\mathscr{r}$ is the distance from the source charge (moving with speed v) to the field point (the centre of the sphere, in this case). Also, keep in mind that, because of the symmetry of the problem, only a spefic component of the magnetic field due to our charged volume element will survive.

The plan is to evaluate this integral one variable at a time so as to also find the magnetic fields at the centre of a (i) current loop and (ii) rotating charged disk.

(c) First, assume that the sphere is to be rotated about the z-axis. Show that charge per unit length λ of our volume element is given by:

$$\begin{aligned}\lambda &= \frac{\mathrm{d}Q}{r\sin\theta\,\mathrm{d}\phi}\\ &= \rho r\,\mathrm{d}\theta\,\mathrm{d}r,\end{aligned}$$

such that:

$$\mathrm{d}B = \frac{\mu_0}{4\pi}\,\omega\lambda\sin^3\theta\,\mathrm{d}\phi.$$

Finally, we will now set $\theta = \pi/2$ (so that out volume element lies in the xy-plane). Then, upon evaluating the ϕ integral, we will have found the magnetic field at the centre of a ring of radius r which corresponds to the origin of our coordinate system. Show that:

$$\boxed{B_{\text{ring}} = \frac{1}{2}\mu_0\lambda\omega.}$$

(d) Now, return to Eq. (1). We still imagine that the sphere is rotating about the z-axis and now the goal is to find the magnetic field due to a rotating charged disk. Show that the charge per unit area of our volume element can be expressed as:

$$\begin{aligned}\sigma_{\text{disk}} &= \frac{\mathrm{d}Q}{r\sin\theta\,\mathrm{d}\phi\,\mathrm{d}r}\\ &= \rho r\,\mathrm{d}\theta,\end{aligned}$$

such that:

$$\mathrm{d}B = \frac{\mu_0}{4\pi}\omega\sigma_{\text{disk}}\sin^3\theta\,\mathrm{d}\phi\,\mathrm{d}r.$$

Now set $\theta = \pi/2$ (so that out volume element lies in the xy-plane). Then, upon evaluating the ϕ and r integrals, we will have found the magnetic field at the centre of disk of radius R. Show that:

$$\boxed{B_{\text{disk}} = \frac{1}{2}\mu_0\sigma_{\text{disk}}\omega R.}$$

(e) Finally, return to Eq. (1) again and evaluate the integrals over r, θ, and ϕ to find the magnetic field at the centre of the spinning sphere. Specially, show that:

$$\boxed{B_{\text{sphere}} = \frac{1}{3}\mu_0\rho\omega R^2.}$$

(f) This part of the problem won't be graded. However, it is worth noting that we can change the order of integration to find the magnetic field at the centre of a uniformly-charged spherical shell. Show that, for this case, the charge per unit area of our volume element can be expressed as:

$$\begin{aligned}\sigma_{\text{shell}} &= \frac{\mathrm{d}Q}{r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi}\\ &= \rho\,\mathrm{d}r,\end{aligned}$$

such that:

$$\mathrm{d}B = \frac{\mu_0}{4\pi}\omega\sigma_{\text{shell}}r\sin^3\theta\,\mathrm{d}\theta\,\mathrm{d}\phi.$$

Completing the integration reveals that the magnetic field at the centre of a uniformly-charged spherical shell spinning with angular speed ω is given by:

$$\boxed{B_{\text{shell}} = \frac{2}{3}\mu_0 \sigma_{\text{shell}} \omega r.}$$

2. We have now seen, perhaps not surprisingly, that spinning charged objects generate magnetic fields. There is a *magnetic dipole moment* of magnitude μ associated with each of these objects that can be calculated using:[1]

$$\mu = \pi \int r^2 \, \mathrm{d}I. \tag{2}$$

You might have already noticed from mechanics that Eq. (2) is similar to how the rotational inertia I_{rot} of a rigid body is calculated:[2]

$$I_{\text{rot}} = \int r^2 \, \mathrm{d}m. \tag{3}$$

Recall also that the angular momentum of a rotating rigid body has magnitude:

$$L = I_{\text{rot}} \omega \tag{4}$$

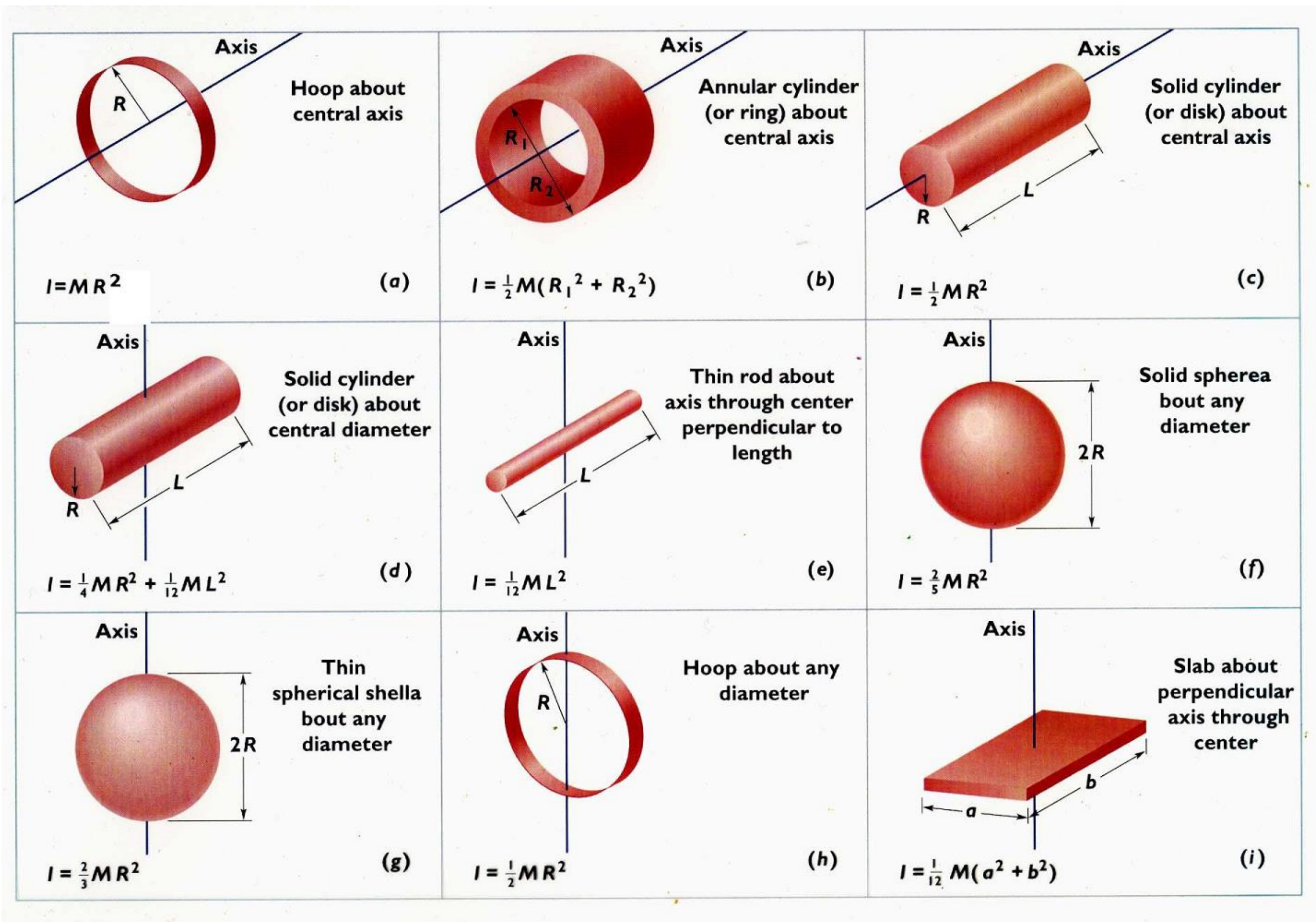
Use Eqs. (2)-(4) to show that the so-called *gyromagnetic ratio* $\gamma = \mu/L$ is given by:

$$\boxed{\gamma = \frac{Q}{2M},}$$

where Q is the total charge of a rotating body and M is its mass. Assume that both the charge and mass densities of the rotating body are uniform. This analysis serves to illustrate the close link between magnetic moments and angular momentum.

[1] Griffiths uses m for the magnetic moment. I will use μ in this assignment to avoid confusion with mass.
[2] Another notational inconvenience! I'll use I for current and I_{rot} for rotational inertia.

3. We could evaluate the integral in Eq. (2) to find the magnetic moments of the ring, disk, spherical shell, and solid sphere that we considered in the first problem. However, that's not necessary because the rotational inertia for these shapes have already been tabulated and we can convert from angular momentum to magnetic moment using the gyromagnetic ratio. Use the following table of rotational inertia to find the magnetic moments of a uniformly-charged spinning (i) ring, (ii) disk, (iii) sphere, and (iv) spherical shell.



4. An electron possesses an intrinsic magnetic moment of magnitude:

$$\mu_s = g_\text{s} \frac{e}{2m} S$$

where S is the the "spin angular momentum". The so-called g-factor $g_\text{s} \approx 2$ and it is a quantum mechanical correction to the classical gyromagnetic ratio derived in question 2. It is conventional to write the magnetic moment as:

$$\mu_s = g_\text{s} \mu_\text{B} \frac{S}{\hbar},$$

where $\mu_\text{B} = e\hbar/(2m)$ is the Bohr magneton and $S = \hbar/2$ is the electron's spin angular momentum.

The reason S is called "spin" is because it is as if the electron were a small charged ball spinning such that it possess angular momentum S and magnetic moment μ. However, as we'll see, we should not take this picture literally. It is just a cartoon used to try to give a classical analogue of the purely quantum mechanical properties of the electron. We should view the electron's magnetic moment and spin angular momentum as intrinsic properties of the electron, in the same way that it has an intrinsic mass m_e and charge e.

To see that the "spin" picture should not be taken literally, we will use the so-called classical electron radius to estimate how fast an electron would need to spin to achieve its observed spin angular momentum and magnetic moment. Again, this is just a toy model. The electron is a point particle with no well-defined radius, however, for the purposes of this calculation, we shall adopt the classical electron radius which is given by:

$$r_\mathrm{e} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_\mathrm{e}c^2} \approx 2.8179 \times 10^{-15}\ \mathrm{m}.$$

(a) Assume the electron can be treated as a uniform ball of mass m_e, charge e, and radius r_e. At what angular speed ω would it need to spin in order to achieve an angular momentum $S = \hbar/2$ and magnetic moment $\mu = \mu_\mathrm{B}$?

(b) How fast, in terms if the speed of light c, would a point on the surface of such a spherical electron be moving? Take the point to be a distance r_e away from the rotation axis.

This analysis should have revealed that a classical spherical electron would have to spin at an unrealistic speed to account for its spin angular momentum and magnetic moment, thus confirming that this picture is not viable. Even if relativistic effects are taken into account, this classical model cannot properly account for the observed spin angular momentum and magnetic moment. Furthermore, both of these quantities are quantized which clearly has no classical analogue.